Categories of holomorphic line bundles on higher dimensional noncommutative complex tori

Hiroshige Kajiura [*]

Yukawa Institute for Theoretical Physics, Kyoto University
Kyoto 606-8502, Japan

Abstract

We construct explicitly noncommutative deformations of categories of holomorphic line bundles over higher dimensional tori. Our basic tools are Heisenberg modules over noncommutative tori and complex/holomorphic structures on them introduced by A. Schwarz. We obtain differential graded (DG) categories as full subcategories of curved DG categories of Heisenberg modules over the complex noncommutative tori. Also, we present the explicit composition formula of morphisms, which in fact depends on the noncommutativity.

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[*]e-mail address: kajiura@yukawa.kyoto-u.ac.jp

Contents

1 Introduction

In this paper, we propose a way to construct differential graded (DG) categories of finitely generated projective modules over higher dimensional noncommutative complex tori. Also, we give explicit examples of this construction as noncommutative deformations of the DG categories of holomorphic line bundles over higher dimensional complex tori.

The motivation of the present paper is to construct an explicit example of the noncommutative deformations of a complex manifold which may be thought of as one of the extended deformations of a complex manifold proposed by Barannikov-Kontsevich [1]. For a n-dimensional complex or a Calabi-Yau manifold M, the extended deformation is defined by a deformation $\epsilon \in \mathfrak{g}^1$ of the Dolbeault operator $\bar{\partial} : \mathfrak{g}^k \to \mathfrak{g}^{k+1}$ such that $(\bar{\partial} + \epsilon)^2 = 0$, where $\mathfrak{g} := \oplus_{k=0}^n \mathfrak{g}^k$ is the graded vector space given by $\mathfrak{g}^k := \oplus_{k=p+q} \Gamma(M, \wedge^p TM \otimes \wedge^q \bar{T} M^*)$. The degree one graded piece consists of $\mathfrak{g}^1 = \Gamma(M, \wedge^2 TM) \oplus \Gamma(M, TM \otimes \bar{T} M^*) \oplus \Gamma(M, \wedge^2 \bar{T} M^*)$. Namely, it defines an extended deformation of the usual complex structure deformation $\epsilon \in \Gamma(M, TM \otimes \bar{T} M^*)$. In particular, the deformation corresponding to $\epsilon \in \Gamma(M, \wedge^2 TM)$ is called the *noncommutative deformation* of the complex manifold M. On the other hand, examples of the models of noncommutative deformation should be constructed so that we can see how it is noncommutative, as in the case of the deformation quantization [26]. A candidate of it may be to consider an algebra deformation of M. Let $V := \oplus_{k=0}^n V^k$ be a graded vector space given by $V^k := \Gamma(M, \wedge^k \bar{T} M)$. This V has a natural graded commutative product $\cdot : V^k \otimes V^l \to V^{k+l}$ together with a differential given by the Dolbeault operator $\bar{\partial} : V^k \to V^{k+1}$. Then, $(V, \bar{\partial}, \cdot)$ forms a DG algebra. A deformation of this DG algebraic structure may describe the noncommutative deformation corresponding to $\epsilon \in \Gamma(M, \wedge^2 TM)$. We can also replace this DG algebra with the DG category of holomorphic vector bundles or coherent sheaves on M, where the DG algebra V should be included in the DG category as the endomorphism algebra of the structure sheaf on M. This algebraic or categorical approach can be thought of as a part of the spirits of *homological mirror symmetry* by Kontsevich [25].

Actually, (topological) string theory suggests considering the graded vector spaces $\mathfrak{g}$ and V; the algebraic structures on their cohomologies are defined as the *closed* [46] *and open* [47] *string amplitudes*, respectively, in the B-twisted topological string theory (B-model). The DG category

above corresponds to physically what is called a *D-brane category* (see [28]); the objects are the D-branes, the morphisms are the open strings between the D-branes in the B-model. Thus, the approach above is physically to construct an open string model instead of the closed string model.

It is natural from viewpoints of both string theory (see [12, 14]) and deformation theory (see [39, 10] and references therein) that DG categories constructed as above should be treated in the category of A_∞-categories, where equivalence between A_∞-categories should be defined by the homotopy equivalence.

Now, let us concentrate on a n-dimensional complex tori $(M := T^{2n}, \bar{\partial})$. It would be easy if its noncommutative deformation corresponding to $\epsilon \in \Gamma(T^{2n}, \wedge^2 TM)$ can be described by the DG algebra $(V, \bar{\partial}, *)$, where $* : V^k \otimes V^l \to V^{k+l}$ is the natural extension of the Moyal product on V^0, the space of functions on T^{2n}, defined by the Poisson bivector $\epsilon \in \Gamma(T^{2n}, \wedge^2 TM)$. However, as far as one identifies homotopy equivalent DG algebras with each other, all these DG algebras turn out to be equivalent, being independent of the noncommutative parameter ϵ. In fact, one can easily show that the DG algebra $(V, \bar{\partial}, *)$ is formal, *i.e.* , homotopy equivalent to a graded algebra on the cohomology $H(V, \bar{\partial})$, and in particular the product on the cohomology $H(V, \bar{\partial})$ is independent of ϵ. These results follow from the fact that one can take a Hodge-Kodaira decomposition of the complex $(V, \bar{\partial})$ so that the harmonic form is closed with respect to the product $*$.

Therefore, in the same way as in the complex one-tori (= real two-tori) case [13, 33, 21, 15], we should include nontrivial vector bundles which are compatible with the complex structure in some sense. In the real two-tori case, one can construct a DG category of holomorphic vector bundles, in the sense of [41], over a noncommutative two-torus [33, 15], where holomorphic vector bundles are described by DG-modules. In particular, the derived category of the DG-category is independent of the noncommutativity parameter $\theta \in \mathbb{R}$ [33]. Though noncommutative deformation of complex tori in this approach is relatively well understood for complex one-tori case, its higher dimensional extension is quite nontrivial and interesting especially from the viewpoint of the extended deformation [1, 18, 11, 2]. However, in this higher dimensional case, a different problem will arise. Even though we start with a DG-module of a holomorphic vector bundle, its noncommutative deformation might not be described by a DG module. There may be several ways to resolve this problem. Our idea in this paper is that we treat the deformed holomorphic vector bundles as *curved differential graded (CDG)*-modules over V. The important point is that even though the deformed ones are not DG modules, the space of morphisms may be equipped with a differential. Namely, in the context of DG categories, the cohomology should be defined not on the objects but on the *morphisms* between the objects. Thus, one may be able to extract finite dimensional graded vector spaces as the cohomologies of the morphisms.

According to such a spirit, we construct DG categories consisting of some of these CDG modules of deformed holomorphic vector bundles on higher dimensional noncommutative tori. We remark that this procedure is just the same as the DG categories of B-twisted topological Landau-Ginzburg model by [19, 20, 43] and also similar to the procedure by FOOO [10] in the mirror dual A-model side. It would also be interesting to construct a triangulated category via the twisted complexes as is done in [43, 44, 4].

Our starting point is based on A. Schwarz's framework of noncommutative supergeometry

[42] and noncommutative complex tori [41, 5]. In [42], a CDG-algebra [35] is re-studied and applied to noncommutative geometry under the name a *Q-algebra*, where modules over a Q-algebra is discussed. On the other hand, in [41], a complex structure is introduced on a real $2n$-dimensional noncommutative torus T_θ^{2n}, and a holomorphic structure on the *Heisenberg modules*, noncommutative analogs of vector bundles, over T_θ^{2n} is defined. Then, our set-up can be thought of as an application of the noncommutative supergeometry [42] to the theory of holomorphic Heisenberg modules [41]. This set-up provides us with explicit descriptions of noncommutative models. Though one of our motivation comes from Fukaya's noncommutative model of Lagrangian foliations on symplectic tori [9] and their mirror dual, our approach in this paper is different from the one since we deal with the Heisenberg modules which are *finitely generated* projective modules over noncommutative tori. For recent papers, see [3] for another approach to noncommutative complex tori and the set-up in [4] which should be closer to ours.

The construction of this paper is as follows. In section 2, we recall the definitions of CDG algebras [35], CDG modules and CDG categories. The notion of modules over a Q-algebra is more general than that of the CDG-modules over a CDG algebra. However, for our purpose, it is enough to consider CDG modules since we begin with Heisenberg modules with a *constant* curvature connections. In section 3, we construct CDG categories of Heisenberg modules over noncommutative tori with complex structures. In particular, we propose a way to obtain DG categories as full subcategories of the CDG categories. Along the general strategy in section 3, we construct CDG categories of holomorphic line bundles over noncommutative complex tori and the DG categories as their full subcategories in section 4. In subsection 4.1, we construct the CDG category on a commutative complex tori. In this case, the CDG category is exactly a DG category. In subsection 4.2, we consider three types of noncommutative deformations of the DG category as CDG categories. Then, we obtain DG categories as the full subcategories of the CDG categories. Furthermore, we present the composition formula of the zero-th cohomologies of the DG categories explicitly. The structure constants of the compositions in fact depend on the noncommutative parameters, which implies that the DG categories or the triangulated/derived categories of them depend on the noncommutative parameters. These results can be thought of as generalizations of complex one-dimensional case [13, 33, 21, 15] and also a complex two-tori case [23, 22] (in the case that the structure constant of the composition is not deformed by the noncommutative parameter). Also, from a string theory or homotopy algebraic point of view, these deformations should corresponds to deformations of an A_∞-structure as weak A_∞-algebras discussed in the context of open-closed homotopy algebras (OCHAs) [17] (see also [29]).

Notations: In this paper, any (graded) vector space stands for the one over the field $k = \mathbb{C}$. We use indices $i, j, \cdots$ for both the ones which run over $1, \cdots, d = 2n$ and the ones which run over $1, \cdots, n$, where n and $d = 2n$ are the complex and the real dimension of a noncommutative torus.

2 CDG algebras, CDG modules and CDG categories

Definition 2.1 ((Cyclic) curved differential graded (CDG) algebra [35]) A *curved differential graded (CDG) algebra* (V, f, d, m) consists of a $\mathbb{Z}$ (or $\mathbb{Z}_2$) graded vector space $V = \oplus_{k \in \mathbb{Z}} V^k$, where V^k is the degree k graded piece, equipped with a degree two element $f \in V^2$, a degree one differential $d : V^k \to V^{k+1}$ and a degree preserving bilinear map $m : V^k \otimes V^l \to V^{k+l}$ satisfying the following relations:

$$d(f) = 0 , \tag{1}$$

$$(d)^2(v) = m(f, v) - m(v, f) , \tag{2}$$

$$dm(v, v') = m(d(v), v') + (-1)^{|v|} m(v, d(v')) , \tag{3}$$

$$m(m(v, v'), v'') = m(v, m(v', v'')) , \tag{4}$$

where $|v|$ is the degree of v, that is, $v \in V^{|v|}$.

Suppose that we have in addition a nondegenerate symmetric inner product

$$\eta : V^k \otimes V^l \to \mathbb{C}$$

of fixed degree $|\eta| \in \mathbb{Z}$ on V. Namely, the η is nondegenerate, nonzero only if $k + l + |\eta| = 0$, and satisfies $\eta(v, v') = (-1)^{kl} \eta(v', v)$ for $v \in V^k$ and $v' \in V^l$. Then, we call (V, f, η, d, m) a *cyclic CDG algebra* if the following conditions hold:

$$\eta(d(v), v') + (-1)^{|v|} \eta(v, d(v')) = 0 , \qquad \eta(m(v, v'), v'') = (-1)^{|v|(|v'| + |v''|)} \eta(m(v', v''), v) .$$

Remark 2.2 A CDG algebra is identified with a weak A_∞-algebra $(V, \{m_k : V^{\otimes k} \to V\}_{k \geq 0})$ with $m_0 = f$, $m_1 = d$, $m_2 = \cdot$ and $m_3 = m_4 = \cdots = 0$. This algebraic structure is what is called a Q-algebra introduced in the framework of noncommutative supergeometry in [42]. Also, a CDG algebra $(V, f, d, \cdot)$ with $f = 0$ is a DG algebra, which is a (strict) A_∞-algebra $(V, \{m_k\}_{k \geq 1})$ with $m_3 = m_4 = \cdots = 0$.

Definition 2.3 (CDG module) A *right CDG module* $(\mathcal{E}, d^\mathcal{E}, m^\mathcal{E})$ over a CDG algebra $(V, -f, d, m)$ is a $\mathbb{Z}$-graded vector space $\mathcal{E}$ equipped with a degree one linear map $d^\mathcal{E} : \mathcal{E} \to \mathcal{E}$ and a right action $m^\mathcal{E} : \mathcal{E} \otimes V \to \mathcal{E}$ satisfying the following condition: for any $v, v' \in V$ and $v^\mathcal{E} \in \mathcal{E}$,

$$(d^\mathcal{E})^2(v^\mathcal{E}) = m^\mathcal{E}(v^\mathcal{E}, f) ,$$

$$d^\mathcal{E} m^\mathcal{E}(v^\mathcal{E}, v) = m^\mathcal{E}(d^\mathcal{E}(v^\mathcal{E}), v) + (-1)^{|\xi|} m^\mathcal{E}(v^\mathcal{E}, d(v)) ,$$

$$m^\mathcal{E}(v^\mathcal{E}, m(v, v')) = m^\mathcal{E}(m^\mathcal{E}(v^\mathcal{E}, v), v') .$$

In particular, if $f = 0$, then $(\mathcal{E}, d^\mathcal{E}, m^\mathcal{E})$ is called a *DG-module* over a DG algebra (V, d, m).

The third condition is nothing but the condition that $\mathcal{E}$ is a (graded) right module over V.

In the category extension of CDG algebras, elements of a CDG algebra turns out to be morphisms in the CDG-category.

Definition 2.4 ((Cyclic) CDG category) A *CDG category* $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C}) = \{a, b, \cdots\}$, a $\mathbb{Z}$-graded vector space $V_{ab} = \oplus_{k \in \mathbb{Z}} V_{ab}^k$ for each two objects a, b and the grading $k \in \mathbb{Z}$, $f_a : \mathbb{C} \to V_{aa}^2$ for each a, a differential $d : V_{ab}^k \to V_{ab}^{k+1}$ and a composition of morphisms $m : V_{bc}^k \otimes V_{ab}^l \to V_{ac}^{k+l}$ satisfying the following relations:

$$d(f_a) = 0 \;, \tag{5}$$

$$(d)^2(v_{ab}) = m(f_b, v_{ab}) - m(v_{ab}, f_a) \;, \tag{6}$$

$$dm(v_{bc}, v_{ab}) = m(d(v_{bc}), v_{ab}) + (-1)^{|v_{bc}|} m(v_{bc}, d(v_{ab})) \;, \tag{7}$$

$$m(m(v_{cd}, v_{bc}), v_{ab}) = m(v_{cd}, m(v_{bc}, v_{ab})) \;, \tag{8}$$

where $|v_{ab}|$ is the $\mathbb{Z}$-grading of v_{ab}, that is, $v_{ab} \in V_{ab}^{|v_{ab}|}$.

Let η be a nondegenerate symmetric inner product of fixed degree $|\eta| \in \mathbb{Z}$ on $V := \oplus_{a,b} V_{ab}$. Namely, for each a and b,

$$\eta : V_{ba}^k \otimes V_{ab}^l \to \mathbb{C} \tag{9}$$

is nondegenerate, nonzero only if $k + l + |\eta| = 0$, and satisfies $\eta(V_{ba}^k, V_{ab}^l) = (-1)^{kl} \eta(V_{ab}^l, V_{ba}^k)$. In this situation, we call a CDG category with inner product η a *cyclic CDG category* $\mathcal{C}$ if the following conditions hold:

$$\eta(dv_{ab}, v_{ab}) + (-1)^{|v_{ab}|} \eta(v_{ab}, dv_{ab}) = 0 \;, \tag{10}$$

$$\eta(m(v_{bc} \otimes v_{ab}), v_{ca}) = (-1)^{|v_{bc}|(|v_{ab}| + |v_{ca}|)} \eta(m(v_{ab} \otimes v_{ca}), v_{bc}) \;. \tag{11}$$

Also, we call a cyclic CDG category $\mathcal{C}$ a *cyclic DG category* if $f_a = 0$ for any $a \in \mathrm{Ob}(\mathcal{C})$.

A CDG category $\mathcal{C}$ consisting of one object only is a CDG algebra. Similarly, for a fixed object $a \in \mathrm{Ob}(\mathcal{C})$, the CDG category structure of $\mathcal{C}$ reduces to a CDG algebra (V_{aa}, f_a, d, m). On the other hand, if the space of morphisms $\mathcal{V} = \oplus_{a,b} V_{ab}$ is thought of as a $\mathbb{Z}$-graded vector space, $(\mathcal{V}, \eta, \oplus_{a \in \mathrm{Ob}(\mathcal{C})} f_a, d, m)$ can be regarded as a cyclic CDG algebra (see also [42]).

Suppose that a CDG category $\mathcal{C}$ has an object $o \in \mathrm{Ob}(\mathcal{C})$ such that (V_{oo}, η, d, m) forms a DG algebra and for any object $a \in \mathrm{Ob}(\mathcal{C})$ there exists a center $\hat{f}_a$ in V_{oo} such that

$$m(f_a, v_{oa}) = m(v_{oa}, \hat{f}_a) \;.$$

Then, $(V_{oa} =: \mathcal{E}_a, d, m)$ can be regarded as a CDG module over the cyclic CDG algebra $(V_{oo}, \eta, -\hat{f}_a, d, m)$.

3 CDG modules and CDG categories on noncommutative tori

3.1 Higher dimensional noncommutative tori

Let us consider an algebra generated by U_i, $i = 1, \cdots, d$, with relations

$$U_j U_k = e^{-2\pi \sqrt{-1} \theta^{jk}} U_k U_j \;, \qquad j, k = 1, \cdots, d \tag{12}$$

for an antisymmetric $d \times d$ matrix $\theta := \{\theta^{jk}\}$. Any element is spanned over $\mathbb{C}$ by elements $U_{\vec{m}}$, $\vec{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, which are defined by

$$U_{\vec{m}} := U_1^{m_1} U_2^{m_2} \ldots U_d^{m_d} e^{\pi \sqrt{-1} \sum_{j<k} m_j m_k \theta^{jk}} \;.$$

The relation between $U_{\vec{m}}$ and $U_{\vec{m}'}$ becomes

$$U_{\vec{m}}U_{\vec{m}'} = e^{\pi\sqrt{-1}\sum_{j,k}m_j\theta^{jk}m_k'}U_{\vec{m}+\vec{m}'} \ . \tag{13}$$

One can represent any element of this algebra as

$$u = \sum_{\vec{m}\in\mathbb{Z}^d} u_{\vec{m}}U_{\vec{m}} \ , \qquad u_{\vec{m}} \in \mathbb{C} \ .$$

For any element u represented as above, an involution $*$ is defined by

$$u^* := \sum_{\vec{m}\in\mathbb{Z}^d} u_{\vec{m}}^* U_{\vec{m}}^* \ ,$$

where $u_{\vec{m}}^*$ is the complex conjugate of $u_{\vec{m}}$ and $U_{\vec{m}}^* := U_{-\vec{m}}$. One can consider a subalgebra T_θ^d such that any element, again represented as $u = \sum_{\vec{m}} u_{\vec{m}}U_{\vec{m}}$, belongs to the Schwartz space $\mathcal{S}(\mathbb{Z}^d)$, that is, the coefficients $\{u_{\vec{m}}\}$ as a function on $\mathbb{Z}^d$ tend to zero faster than any power of $||\vec{m}||$. This algebra T_θ^d is in fact a C^*-algebra and called (the smooth version of) a *noncommutative torus* [36, 24].

There is a canonical normalized *trace* on T_θ^d specified by the rule

$$\mathrm{Tr}(u) = u_{\vec{m}=0} \ , \qquad u = \sum_{\vec{m}} u_{\vec{m}}U_{\vec{m}} \ . \tag{14}$$

For $\theta = 0$ we can realize the algebra T_θ^d as an algebra of functions on a d-dimensional torus T^d. Then the trace (14) corresponds to an integral over T^d provided the volume of T^d is one.

In order to define a connection on a module E over noncommutative torus T_θ^d we shall first define a natural Lie algebra of shifts $\mathcal{L}_\theta$ acting on T_θ^d. The shortest way to define this Lie algebra is by specifying a basis consisting of derivations $\delta_j : T_\theta^d \to T_\theta^d$, $j = 1, \ldots, d$, satisfying

$$\delta_j(U_{\vec{m}}) = 2\pi\sqrt{-1}m_j U_{\vec{m}} \ . \tag{15}$$

For the multiplicative generators U_j the above relation reads as

$$\delta_j U_k = 2\pi\sqrt{-1}\delta_{jk}U_k \ .$$

This derivations then span a d-dimensional abelian Lie algebra (over $\mathbb{C}$) that we denote $\mathcal{L}_\theta$.

A *connection* on a (right) module E over T_θ^d is a set of operators $\nabla_X : E \to E$, $X \in \mathcal{L}_\theta$ depending linearly on X and satisfying

$$\nabla_X(\xi \cdot u) = \nabla_X(\xi) \cdot u + \xi \cdot X(u)$$

for any $\xi \in E$ and $u \in T_\theta^d$. In general, connections whose curvature equals the identity endomorphism times a numerical tensor are called constant curvature connections. We express the *constant curvature* of a constant curvature connection $\nabla_i := \nabla_{\delta_i}$, $i = 1, \cdots, d$, as

$$[\nabla_i, \nabla_j] = -2\pi\sqrt{-1}F_{ij} \cdot \mathbf{1}_{\mathrm{End}_{T_\theta^d}(E)} \ , \qquad F_{ij} = -F_{ji} \in \mathbb{R} \ . \tag{16}$$

On a noncommutative torus, one can construct a class of finitely generated projective modules called *Heisenberg modules* (see [36, 24]). In fact, on any Heisenberg module there exists

6

a constant curvature connection. They play a special role. It was shown by Rieffel ([36]) that if the matrix θ^{ij} is irrational in the sense that at least one of its entries is irrational then any projective module over T_θ^d is isomorphic to a direct sum of Heisenberg modules.

Heisenberg modules are applied to discuss the Morita equivalence of noncommutative tori. A noncommutative tori T_θ^d is Morita equivalent to $T_{\theta'}^d$ if [37] and only if [40] $\theta' = g(\theta)$, $g \in SO(d, d, \mathbb{Z})$ (for more recent papers, see [45, 31, 7]). Here, the $SO(d, d, \mathbb{Z})$ action on the space of skew symmetric matrices in $Mat_d(\mathbb{R})$ is defined by

$$g(\theta) := (\mathcal{A}\theta + \mathcal{B})(\mathcal{C}\theta + \mathcal{D})^{-1} , \quad g := \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in SO(d, d, \mathbb{Z}) ,$$

where $SO(d, d, \mathbb{Z}) := \{g \in Mat_{2n}(\mathbb{Z}) \mid g^t J g = J\}$ for $J := \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. To establish this Morita equivalence, one may construct a T_θ^d-$T_{g(\theta)}^d$ *Morita equivalence bimodule*, denoted by $P_{\theta\text{-}g(\theta)}$ (see [36, 37]). One can in fact construct the Morita equivalence bimodule $P_{\theta\text{-}g(\theta)}$ for any $g \in SO(d, d, \mathbb{Z})$ as a left Heisenberg module E over T_θ^d. In this case, the algebra $\mathrm{End}_{T_\theta^d}(E)$, the algebra of endomorphisms of E which commute with the left action of T_θ^d, coincides with the noncommutative torus $T_{g(\theta)}^d$. This implies that one can construct a $T_{g(\theta)}^d$-T_θ^d Morita equivalence bimodule $P_{g(\theta)\text{-}\theta}$ as a right Heisenberg module over T_θ^d. We denote it by $E_{g,\theta}$; we have $\mathrm{End}_{T_\theta^d}(E_{g,\theta}) \simeq T_{g(\theta)}^d$. Also, the T_θ^d-$T_{g(\theta)}^d$ Morita equivalence bimodule is given by the right Heisenberg module $E_{g^{-1}, g(\theta)}$. On $T_{g(\theta)}^d$, a *trace* $\mathrm{Tr}_{T_{g(\theta)}^d} : T_{g(\theta)}^d \to \mathbb{C}$ and *derivations* $\delta_i : T_{g(\theta)}^d \to T_{g(\theta)}^d$, $i = 1, \cdots, d$, are defined by appropriate rescaling of those for T_θ^d as

$$\mathrm{Tr}_{T_{g(\theta)}^d}(u) = \sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}\, u_{\vec{m}=0} \qquad u := \sum_{\vec{m} \in \mathbb{Z}^n} u_{\vec{m}} Z_{\vec{m}} \in T_{g(\theta)}^d \tag{17}$$

and

$$\delta_j(Z_{\vec{m}}) = \frac{2\pi\sqrt{-1}m_j}{\sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}} Z_{\vec{m}} ,$$

where $g = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ and $Z_1, \cdots, Z_d$ are the generators of $T_{g(\theta)}^d$ with relations $Z_j Z_k = e^{-2\pi\sqrt{-1}(g\theta)^{jk}} Z_k Z_j$, $j, k = 1, \cdots, d$. For $X \in \mathcal{L}_\theta$, a linear map $\nabla : T_{g(\theta)}^d \otimes \mathcal{L}_\theta \to T_{g(\theta)}^d$ is defined by extending linearly

$$\nabla_{\delta_i}(u) := \delta_i(u) , \quad i = 1, \cdots, d , \qquad u \in T_{g(\theta)}^d . \tag{18}$$

When we have a T_θ^d-$T_{\theta'}^d$ Morita equivalence bimodule, one can consider the following *tensor product* (see [40, 42]):

$$E_{g_a,\theta} \otimes_{T_\theta^d} P_{\theta\text{-}g(\theta)} \simeq E_{g_a g^{-1}, g(\theta)} \tag{19}$$

for a Heisenberg module $E_{g_a,\theta}$ with any $g_a \in SO(d, d, \mathbb{Z})$, where the tensor product $\otimes_{T_\theta^d}$ is defined by the standard tensor product $\otimes$ over $\mathbb{C}$ with the identification $(\xi_a \cdot u) \otimes p \sim \xi_a \otimes (u \cdot p)$ for any $\xi_a \in E_{g_a,\theta}$, $u \in T_\theta^d$ and $p \in P_{\theta\text{-}g(\theta)}$. Let us denote $\theta_a := g_a(\theta)$. For a right Heisenberg module $E_{g_a,\theta}$ and a Morita equivalence bimodule $P_{\theta_b\text{-}\theta_a}$ with a $SO(d, d, \mathbb{Z})$ element g_b, the existence of the tensor product (19) implies that we have the following tensor product:

$$P_{\theta_b\text{-}\theta_a} \otimes_{T_{\theta_a}^d} E_{g_a,\theta} \simeq E_{g_b,\theta} .$$

7

Thus, we may identify $P_{\theta_b \text{-}\theta_a}$ with the space of homomorphisms from $E_{g_a,\theta}$ to $E_{g_b,\theta}$. Hereafter we write

$$\text{Hom}(E_{g_b,\theta}, E_{g_a,\theta}) := P_{g_b(\theta)\text{-}g_a(\theta)} \ .$$

On a Morita equivalence $T^d_{\theta_b}$-$T^d_{\theta_a}$ bimodule $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, we define a connection ∇ : $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \otimes \mathcal{L}_\theta \to \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ by a linear map satisfying the following relation:

$$\nabla_X(u_b \cdot \xi) = \nabla_X(u_b) \cdot \xi + u_b \cdot \nabla_X(\xi) \ , \quad \nabla_X(\xi \cdot u_a) = \nabla_X(\xi) \cdot u_a + \xi \cdot \nabla_X(u_a) \ , \quad u_a \in T^d_{\theta_a} \ , \quad u_b \in T^d_{\theta_b} \ ,$$

where $\nabla_X(u_b)$ and $\nabla_X(u_a)$ are defined by eq.(18). Since these Morita equivalence bimodules are Heisenberg modules, they can be equipped with constant curvature connections.

A Heisenberg module $E_{g,\theta}$ attached to an element $g \in SO(d,d,\mathbb{Z})$ as above is called a *basic module*. Since any Heisenberg module is constructed by a direct sum of basic modules, in this paper we concentrate on categories of basic modules. Let $\text{Ob} := \{a, b, \cdots\}$ be a collection of labels and consider a map $g : \text{Ob} \to SO(d,d,\mathbb{Z})$, $g(a) := g_a \in SO(d,d,\mathbb{Z})$ for $a \in \text{Ob}$. For the collection of Heisenberg modules $\{E_{g_a,\theta} \mid a \in \text{Ob}\}$, assume we have an associative product

$$m : \text{Hom}(E_{g_b,\theta}, E_{g_c,\theta}) \otimes \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \to \text{Hom}(E_{g_a,\theta}, E_{g_c,\theta})$$

for any $a, b, c \in \text{Ob}$. Namely, for any $a, b, c, d \in \text{Ob}$ and $\xi_{ab} \in \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, $\xi_{bc} \in \text{Hom}(E_{g_b,\theta}, E_{g_c,\theta})$, $\xi_{cd} \in \text{Hom}(E_{g_c,\theta}, E_{g_d,\theta})$, we assume that the product m satisfies

$$m(m(\xi_{cd}, \xi_{bc}), \xi_{ab}) = m(\xi_{cd}, m(\xi_{bc}, \xi_{ab})) \ . \tag{20}$$

Such a product $m : \text{Hom}(E_{g_b,\theta}, E_{g_c,\theta}) \otimes \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \to \text{Hom}(E_{g_a,\theta}, E_{g_c,\theta})$ is essentially the tensor product; m is constructed by fixing a map inducing the isomorphism

$$\text{Hom}(E_{g_b,\theta}, E_{g_c,\theta}) \otimes_{T^d_{\theta_b}} \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \simeq \text{Hom}(E_{g_a,\theta}, E_{g_c,\theta}) \ .$$

There exists a choice of the map so that the associativity holds (see [42]).

For $a \in \text{Ob}$, suppose that a constant curvature connection ∇_a is defined on $E_{g_a,\theta}$. Also, for $b \in \text{Ob}$, $a \neq b$, define a constant curvature connection $\nabla : \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \otimes \mathcal{L}_\theta \to \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ whose constant curvature $F_{ab} := \{F_{ab,ij}\}_{i,j=1,\cdots,d}$ is defined by

$$F_{ab,ij} \cdot \xi_{ab} := \frac{\sqrt{-1}}{2\pi} [\nabla_i, \nabla_j](\xi_{ab}) \ , \qquad F_{ab,ij} = -F_{ab,ji} \in \mathbb{R}$$

for any $\xi \in \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$. Then, a constant curvature connection $\nabla_b : E_{g_b,\theta} \otimes \mathcal{L}_\theta \to E_{g_b,\theta}$ can be induced as follows [40, 42]:

$$\nabla_b(m(\xi_{ab}, \xi_a)) := m(\nabla(\xi_{ab}), \xi_a) + m(\xi_{ab}, \nabla_a(\xi_a))$$

for any $\xi_a \in E_{g_a,\theta}$ and $\xi_{ab} \in \text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, where the relation between the curvatures of $E_{g_a,\theta}$, $E_{g_b,\theta}$ and that of $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ is given by

$$F_b - F_a = F_{ab} \ .$$

Thus, repeating this procedure leads to the following category $\mathcal{C}^{pre}_{\theta,E}$:

Definition 3.1 For a noncommutative torus T_θ^d, let $\mathrm{Ob} := \{a, b, \cdots\}$ be a collection of labels and E a map from Ob to the space of basic modules with constant curvature connections; for $a \in \mathrm{Ob}$, we denote $E(a) = (E_{g_a,\theta}, \nabla_a) =: E_a$. A *category* $\mathcal{C}_{\theta,E}^{pre}$ is defined by the following data.

- The collection of objects is

$$\mathrm{Ob}(\mathcal{C}_{\theta,E}^{pre}) := \mathrm{Ob} .$$

Each object $a \in \mathrm{Ob}$ is associated with a basic module with a constant curvature connection E_a whose constant curvature is denoted by a skewsymmetric matrix $F_a \in Mat_{2n}(\mathbb{R})$.

- For any $a, b \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{pre})$, the space of morphisms is

$$\mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) := \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) ,$$

which is equipped with a constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b)$ with its constant curvature $F_{ab} = F_b - F_a$.

- For any $a, b, c \in \mathrm{Ob}$, there exists an associative product (eq.(20))

$$m : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(b, c) \otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, c) .$$

- For any $a, b, c \in \mathrm{Ob}$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(b, c)$, the Leibniz rule holds:

$$\nabla m(\xi_{bc}, \xi_{ab}) = m(\nabla(\xi_{bc}), \xi_{ab}) + m(\xi_{bc}, \nabla(\xi_{ab})) . \tag{21}$$

- For any $a \in \mathrm{Ob}$, a trace $\mathrm{Tr}_a : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, a) \to \mathbb{C}$ is given by eq.(17):

$$\mathrm{Tr}_a(u) = \sqrt{|\det(\mathcal{C}_a \theta + \mathcal{D}_a)|}\, u_{\vec{m}=0} \qquad u := \sum_{\vec{m} \in \mathbb{Z}^n} u_{\vec{m}} Z_{\vec{m}} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, a)$$

for $g_a = \left(\begin{smallmatrix} \mathcal{A}_a & \mathcal{B}_a \\ \mathcal{C}_a & \mathcal{D}_a \end{smallmatrix}\right)$. In particular, for any $a, b \in \mathrm{Ob}$, $\mathrm{Tr}_a\, m : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(b, a) \otimes \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) \to \mathbb{C}$ gives a nondegenerate bilinear map such that

$$\mathrm{Tr}_a\, m(\xi_{ba}, \xi_{ab}) = \mathrm{Tr}_b\, m(\xi_{ab}, \xi_{ba}) , \qquad \xi_{ab} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) , \quad \xi_{ba} \in \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(b, a) . \tag{22}$$

The last identity (22) together with the nondegeneracy of $\mathrm{Tr}_a\, m$ is a typical property of Morita equivalence bimodules (see [36, 24], and for noncommutative two-tori case [15]).

3.2 Noncommutative complex tori and CDG structures on them

Let us consider a complex structure on the noncommutative torus T_θ^{2n} as introduced by A. Schwarz [41]. We take a different notation which fits our arguments, though it is equivalent to the one in [41]. When we define a complex structure on a commutative torus T^{2n}, we may take a $\mathbb{C}$-valued n by n matrix $\tau = \{\tau_j^i\}$, $i, j = 1, \cdots, n$, whose imaginary part $\tau_I := \mathrm{Im}(\tau)$ is positive definite. A commutative complex torus is then described by $\mathbb{C}^n/(\mathbb{Z}^n + \tau^t \mathbb{Z}^n)$, where τ^t is the transpose of τ. The complex coordinates of $\mathbb{C}^n$ are given by $(z_1, \cdots, z_n)$, $z^i = x^i + \sum_j y^j \tau_j^i$, $i = 1, \cdots, n$. The corresponding Dolbeault operator $\bar{\partial}$ is given by

$$\bar{\partial} = \sum_{i=1}^n d\bar{z}^i \frac{\partial}{\partial \bar{z}^i} , \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2\sqrt{-1}} \sum_{j=1}^n \left(((\tau_I)^{-1}\tau)_i^j \frac{\partial}{\partial x^j} - ((\tau_I)^{-1})_i^j \frac{\partial}{\partial y^j} \right) ,$$

9

where we denote $\mathrm{Im}(\tau) =: \tau_I$ which is by definition positive definite.

Based on these formula, for a noncommutative torus T_θ^{2n} and a fixed complex structure τ, let us define $\bar{\partial}_i \in \mathcal{L}_\theta$, $i = 1, \cdots, n$, by

$$\bar{\partial}_i := \frac{1}{2\sqrt{-1}} \sum_{j=1}^n \left(((\tau_I)^{-1}\tau)_i^j \delta_j - ((\tau_I)^{-1})_i^j \delta_{n+j} \right) \ .$$

Also, for $E_a := (E_{g_a,\theta}, \nabla_a)$, a Heisenberg module $E_{g_a,\theta}$ over T_θ^{2n} with a constant curvature connection $\nabla_{a,i}$, $i = 1, \cdots, 2n$, define a *holomorphic structure* $\bar{\nabla}_{a,i} : E_{g_a,\theta} \to E_{g_a,\theta}$, $i = 1, \cdots, n$, by

$$\bar{\nabla}_{a,i} := \frac{1}{2\sqrt{-1}} \sum_{j=1}^n \left(((\tau_I)^{-1}\tau)_i^j \nabla_{a,j} - ((\tau_I)^{-1})_i^j \nabla_{a,n+j} \right) \ . \tag{23}$$

For each pair (E_a, E_b), we define a *holomorphic structure* $\bar{\nabla}_i : \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \to \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$, $i = 1, \cdots, n$, by the same formula:

$$\bar{\nabla}_i := \frac{1}{2\sqrt{-1}} \sum_{j=1}^n \left(((\tau_I)^{-1}\tau)_i^j \nabla_j - ((\tau_I)^{-1})_i^j \nabla_{n+j} \right) \ . \tag{24}$$

Let Λ be the Grassmann algebra generated by $d\bar{z}^1, \cdots, d\bar{z}^n$ of degree one. Namely, they satisfy $d\bar{z}^i d\bar{z}^j = -d\bar{z}^j d\bar{z}^i$ for any $i, j = 1, \cdots, n$, so in particular $(d\bar{z}^i)^2 = 0$. These generators are thought of as the formal basis of the antiholomorphic one forms on the complex noncommutative torus T_θ^{2n}. By Λ^k we denote the degree k graded piece of Λ. The graded vector space $V := T_\theta^{2n} \otimes \Lambda$ is then thought of as the space of the smooth antiholomorphic forms on the complex noncommutative torus T_θ^{2n}, which also has the the graded piece decomposition:

$$V = \oplus_{k=0}^n V^k \ .$$

Any element in V^k can be written as

$$v = \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{i_1, \cdots, i_k} v_{\vec{m}; i_1 \cdots i_k} U_{\vec{m}} \cdot (d\bar{z}^{i_1} \cdots d\bar{z}^{i_k}) \ ,$$

where $v_{\vec{m}; i_1 \cdots i_k} \in \mathbb{C}$ is skewsymmetric with respect to the indices $i_1 \cdots i_k$. A product $m : V^k \otimes V^l \to V^{k+l}$ is defined naturally by combining the product on T_θ^{2n} with the one on the Grassmann algebra Λ, and then (V, m) forms a graded algebra. One can define a differential $d : V^k \to V^{k+1}$,

$$d := \sum_{i=1}^n d\bar{z}^i \cdot \bar{\partial}_i \ ,$$

which satisfies the Leibniz rule with respect to the product m.

An inner product $\eta : V^k \otimes V^l \to \mathbb{C}$ of degree $-n$ is defined by the composition of the product m with a trace $\int_{T_\theta^{2n}} : V \to \mathbb{C}$:

$$\eta = \int_{T_\theta^{2n}} m \ , \qquad \int_{T_\theta^{2n}} v = v_{\vec{m}=0; i_1 \cdots i_k} \epsilon_{1 \cdots n}^{i_1 \cdots i_k} \ .$$

Here ϵ is defined by

$$\epsilon^{i_1 \cdots i_k}_{1 \cdots n} = \begin{cases} 0 & (k \neq n) \\ \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \delta^{i_1}_{\sigma(1)} \cdots \delta^{i_n}_{\sigma(n)} & k = n \ , \end{cases}$$

where $\epsilon(\sigma)$ is the signature of the permutation of n elements $\sigma \in \mathfrak{S}_n$. Namely, $\int_{T^d_\theta} : V^k \to \mathbb{C}$ is thought of as the integration of differential forms over T^{2n}_θ, as an extension of the trace map $\mathrm{Tr} : T^{2n}_\theta \to \mathbb{C}$ in eq.(14), and hence gives a nonzero map only if $k = n$.

Lemma 3.2 (V, η, d, m) *forms a cyclic DG algebra.* ∎

For $E_a := (E_{g_a, \theta}, \nabla_a)$ a Heisenberg module over T^{2n}_θ with a constant curvature connection, we lift E_a to a $\mathbb{Z}$-graded right module $\mathcal{E}_a := E_{g_a, \theta} \otimes \Lambda$ over V. We denote by $m_a : \mathcal{E}_a \otimes V \to \mathcal{E}_a$ the right action of V on $\mathcal{E}_a$. The connection $\nabla_a : E_{g_a, \theta} \otimes \mathcal{L}_\theta \to E_{g_a, \theta}$ is then lifted to a degree one linear map $d_a : \mathcal{E}_a \to \mathcal{E}_a$ defined by

$$d_a := \sum_{i=1}^{n} d\bar{z}^i \cdot \bar{\nabla}_{a,i} \ ,$$

where $\bar{\nabla}_{a,i}$ is the holomorphic structure (23). This d_a is not a differential in general. Namely, the graded module has its curvature:

$$(d_a)^2 v^{\mathcal{E}_a} = \hat{f}_a \cdot v^{\mathcal{E}_a} \ , \qquad \hat{f}_a := -\pi \sqrt{-1} \left(d\bar{z}^t \tau_I^{-1} \right) \begin{pmatrix} \tau & -\mathbf{1}_n \end{pmatrix} F_a \begin{pmatrix} \tau^t \\ -\mathbf{1}_n \end{pmatrix} \left(\tau_I^{t,-1} d\bar{z} \right) \in \Lambda^2$$

for any $v^{\mathcal{E}_a} \in \mathcal{E}_a$, where $d\bar{z}^t := (\bar{z}^1, \cdots, \bar{z}^n)$. This d_a defines a differential on $\mathcal{E}_a$, that is, $\hat{f}_a = 0$ if and only if

$$\begin{pmatrix} \tau & -\mathbf{1}_n \end{pmatrix} F_a \begin{pmatrix} \tau^t \\ -\mathbf{1}_n \end{pmatrix} = 0 \ .$$

In this case, $(\mathcal{E}_a, d_a, m_a)$ forms a DG module over V. In the commutative case $(\theta = 0)$, this condition on F_a is nothing but the condition that the corresponding (line) bundle is holomorphic, *i.e.*, the curvature is of $(1,1)$-form with respect to the complex structure defined by τ. However, for general θ, $\hat{f}_a$ can not be zero even if it is zero when θ is set to be zero.

On the other hand, since $\hat{f}_a \in \Lambda^2 \subset V^2$ is a center in V with respect to the product m, $(d)^2(v) = m(\hat{f}_a, v) - m(v, \hat{f}_a) \ (= 0)$ holds and then $(V, \eta, -\hat{f}_a, d, m)$ forms a cyclic CDG algebra. Thus:

Lemma 3.3 $(\mathcal{E}_a, \hat{f}_a, d_a, m_a)$ *forms a CDG module over the cyclic CDG algebra* $(V, \eta, -\hat{f}_a, d, m)$. ∎

We call this $\hat{f}_a \in V^2$ the *potential two-form* of $\mathcal{E}_a$.

Now, for a category $\mathcal{C}^{pre}_{\theta, E}$ given in Definition 3.1, we construct a CDG category $\mathcal{C}_{\theta, \tau, E} =: \mathcal{C}$.

Definition 3.4 For a fixed category $\mathcal{C}^{pre}_{\theta, E}$, a *category* $\mathcal{C}$ is defined as follows.
- The collection of objects is
$$\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob} \ ,$$

where any object $a \in$ Ob is associated with a CDG module $(\mathcal{E}_a, \hat{f}_a, d_a, m_a)$ over the CDG algebra $(V, \eta, -\hat{f}_a, d, m)$ corresponding to E_a as in Lemma 3.3.

- For any $a, b \in \mathrm{Ob}(\mathcal{C})$, the space of morphisms is the graded vector space

$$\mathrm{Hom}_{\mathcal{C}}(a, b) := \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta, E}}(a, b) \otimes \Lambda =: V_{ab} = \oplus_{k=1}^{n} V^k_{ab} ,$$

which is equipped with a degree one linear map $d : V^k_{ab} \to V^{k+1}_{ab}$,

$$d := \sum_{i=1}^{n} d\bar{z}^i \bar{\nabla}_i ,$$

where $\bar{\nabla}_i$ is the holomorphic structure (24) corresponding to the constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta, E}}(a, b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta, E}}(a, b)$.

- For any $a, b, c \in \mathrm{Ob}(\mathcal{C})$, an associative product $m : V^l_{bc} \otimes V^k_{ab} \to V^{k+l}_{ac}$ is given by the lift of the product m on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta, E}}(*, *)$.

- For any two objects $a, b \in \mathrm{Ob}(\mathcal{C})$, a nondegenerate graded symmetric inner product $\eta : V_{ba} \otimes V_{ab} \to \mathbb{C}$ of degree $-n$ is defined by

$$\eta = \int_{T^{2n}_{\theta_a}} m , \qquad m : V_{ba} \otimes V_{ab} \to V_{aa} .$$

Here $\int_{T^{2n}_{\theta_a}} : V_{aa} \to \mathbb{C}$ is defined by

$$\int_{T^{2n}_{\theta_a}} v = \sqrt{|\det(\mathcal{C}_a \theta + \mathcal{D}_a)|} \, v_{\vec{m}=0; i_1 \cdots i_k} \epsilon^{i_1 \cdots i_k}_{1 \cdots n} \qquad v := \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{i_1, \cdots, i_k} v_{\vec{m}; i_1 \cdots i_k} U_{\vec{m}} \cdot (d\bar{z}^{i_1} \cdots d\bar{z}^{i_k}) \in V_{aa}$$

for $g_a = \begin{pmatrix} \mathcal{A}_a & \mathcal{B}_a \\ \mathcal{C}_a & \mathcal{D}_a \end{pmatrix}$ as an extension of the trace map $\mathrm{Tr}_a \to \mathbb{C}$.

Due to the Leibniz rule (21), it is clear that $d : V^k_{ab} \to V^{k+1}_{ab}$ is a derivation:

$$dm(v_{bc} \otimes v_{ab}) = m(d(v_{bc}) \otimes v_{ab}) + (-1)^{|v_{bc}|} m(v_{bc} \otimes d(v_{ab})) . \tag{25}$$

Let us define $\hat{f}_{ab} \in \Lambda^2$ by

$$\hat{f}_{ab} := d^2 , \qquad d : V^k_{ab} \to V^{k+1}_{ab} .$$

Then, the Leibniz rule (25) and $F_{ab} = F_b - F_a$ imply $\hat{f}_{ab} = \hat{f}_b - \hat{f}_a$. For each $a \in \mathrm{Ob}(\mathcal{C})$, let $f_a := \hat{f}_a \cdot \mathbf{1}_a \in V_{aa}$, where $\mathbf{1}_a$ is the identity in $T^{2n}_{\theta_a}$. The following is the main claim of this paper.

Proposition 3.5 (Cyclic DG category of holomorphic vector bundles)

(i)   *For a given category $\mathcal{C}^{pre}_{\theta, E}$ in Definition 3.1, $\mathcal{C} := \mathcal{C}_{\theta, \tau, E}$ forms a cyclic CDG category.*

(ii)   *Let $\mathcal{C}^{\hat{f}}$ be the full subcategory of $\mathcal{C}$ such that any $a \in \mathrm{Ob}(\mathcal{C}^{\hat{f}}) \subset \mathrm{Ob}(\mathcal{C})$ is a set of CDG modules over a cyclic CDG algebra $(V, \eta, -\hat{f}, d, m)$ for a fixed $\hat{f} \in \Lambda^2 \subset V^2$. Then $\mathcal{C}^{\hat{f}}$ forms a cyclic DG category.*

$\blacksquare$

This implies that one can construct a kind of DG categories of holomorphic vector bundles over a noncommutative tori.

4 Three examples of noncommutative deformations

Now, we construct examples of various noncommutative deformations of the DG categories of Heisenberg modules described by CDG modules over a cyclic CDG algebra of a noncommutative torus. The set-up given in the previous subsection allows us to deform both the complex structure τ and the noncommutativity θ or either of them. In this paper, starting from a commutative ($\theta = 0$) n dimensional complex torus with the standard complex structure $\tau = \sqrt{-1}\mathbf{1}_n$ in subsection 4.1, we deform the noncommutative parameter θ with preserving the standard complex structure in subsection 4.2. Also, we give the composition formula on the zero-th cohomologies of the DG categories explicitly. We show that the structure constants of the compositions in fact depends on θ.

4.1 Commutative case

Let us begin with the commutative torus $T^{2n} := T^{2n}_{\theta=0}$. The generators $U_1, \cdots, U_n, U_{\bar{1}} := U_{n+1}, \cdots, U_{\bar{n}} := U_{2n}$ then commute with each other. The arguments in subsection 3.2 show that it is enough to construct a category $\mathcal{C}^{pre}_{\theta=0,E}$ in order to construct a cyclic CDG category $\mathcal{C}$.

A category $\mathcal{C}^{pre}_{\theta=0,E}$ is constructed as follows. Any object $a \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta=0,E})$ is associated with a pair $E_a := (E_{g_a,\theta=0}, \nabla_a)$ of basic module $E_{g_a,\theta=0}$ with a constant curvature connection ∇_a. The basic module is defined by

$$E_{g_a,\theta=0} := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a fixed nondegenerate symmetric matrix $A_a \in Mat_n(\mathbb{Z})$, where $g_a \in SO(d,d,\mathbb{Z})$ is given by

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_a & \mathbf{1}_{2n} \end{pmatrix} , \qquad F_a := \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} .$$

The right action of T^{2n} on $E_{g_a,\theta=0}$ is defined by specifying the right action of each generator; for $\xi_a \in E_{g_a,\theta=0}$, it is given by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu)e^{2\pi\sqrt{-1}(x_i+(A_a^{-1}\mu)_i))} ,$$
$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + A_a^{-1}t_i; \mu - t_i) , \qquad i = 1, \cdots, n ,$$

where $x := (x_1 \cdots x_n)^t \in \mathbb{R}^n$, $\mu \in \mathbb{Z}^n/A_a\mathbb{Z}^n$ and $t_i \in \mathbb{R}^n$ is defined by $(t_1 \cdots t_n) = \mathbf{1}_n$. A constant curvature connection $\nabla_a : E_{g_a,\theta=0} \otimes \mathcal{L}_\theta \to E_{g_a,\theta=0}$ is given by

$$(\nabla_{a,1} \cdots \nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & \\ & -A_a \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} , \tag{26}$$

where $\partial_x := \left(\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \right)^t$, and the curvature (defined by eq.(16)) is F_a above. The generators of the endomorphism algebra is the same as $U_i, U_{\bar{i}}$:

$$(Z_i\xi_a)(x;\mu) = e^{2\pi\sqrt{-1}(x_i+(A_a^{-1}\mu)_i))}\xi_a(x;\mu) ,$$
$$(Z_{\bar{i}}\xi_a)(x;\mu) = \xi_a(x + A_a^{-1}t_i; \mu - t_i) , \qquad i = 1, \cdots, n .$$

Namely, the endomorphism algebra also forms a commutative torus T^{2n}.

13

This $E_a := (E_{g_a}, \nabla_a)$ is lifted to a CDG module $(\mathcal{E}_a, \hat{f}_a, d_a, m_a)$ over the cyclic CDG algebra $(V = T^{2n} \otimes \Lambda, \eta, -\hat{f}_a, d, m)$ by the procedure in the previous subsection, where the complex structure is taken to be the standard one: $\tau = \sqrt{-1}\mathbf{1}_n$. Then, one obtains $(d_a)^2 = \hat{f}_a = 0$, that is, $\mathcal{E}_a$ in fact forms a DG-module corresponding to a *holomorphic line* bundle.

For any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta=0,E})$, the space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$ is defined as follows. If $A_{ab} := A_b - A_a$ is nondegenerate, then it is again the Schwartz space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$. For $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$, the right action of T^{2n}, generated by U_i and $U_{\bar{i}}$, and the left action of T^{2n}, generated by Z_i and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu)e^{2\pi\sqrt{-1}(x_i+(A_{ab}^{-1}\mu)_i))} \ ,$$
$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i; \mu - t_i) \ ,$$
$$(Z_i\xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}(x_i+(A_{ab}^{-1}\mu)_i))}\xi_{ab}(x;\mu) \ ,$$
$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i; \mu - t_i)$$

for $i = 1, \cdots, n$, where $\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n$. In fact, all these generators U_i, $U_{\bar{i}}$, Z_i and $Z_{\bar{i}}$ commute with each other.

On the other hand, if $A_a = A_b$, we define $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) := T^{2n}$, on which the left and right actions of T^{2n} are defined just by the commutative product on T^{2n}.

In general, the way of constructing the space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) = \mathrm{Hom}(E_{g_a,\theta=0}, E_{g_b,\theta=0})$ depends on the rank of $A_{ab} := A_b - A_a$. In rank n case (nondegenerate case), one has $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) := \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$ and in rank 0 case, one has $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) := T^{2n}$ as above. In rank $1 < r < n$ case, we should combine these two constructions with each other appropriately. In order to avoid such case-by-case arguments, in this paper we assume that A_{ab} is nondegenerate for any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta=0,E})$ such that $a \neq b$.

The constant curvature connection $\nabla_i : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$, $i = 1, \cdots, 2n$, is given by

$$(\nabla_1 \cdots \nabla_{2n})^t := \begin{pmatrix} \mathbf{1}_n & \\ & -A_{ab} \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a \neq b$, and if $a = b$, it is defined by the derivation ∇ on the noncommutative torus $T^{2n}_{\theta_a} = T^{2n}_{\theta_b}$ in eq.(18) with $\theta_a = \theta_b = 0$.

For $a, b, c \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta=0,E})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(b,c)$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(b,c) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,c)$ is given as follows:

For $a = b$, it is the right action of T^{2n} on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(b,c)$.

For $b = c$, it is the left action of T^{2n} on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$.

For $a = c$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(b,a) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b) \to T^{2n}$ is given by

$$m(\xi_{ba}, \xi_{ab})(x,\rho) = \sum_{\vec{m}\in\mathbb{Z}^n} \sum_{\mu\in\mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\vec{m}} \int_{\mathbb{R}^n} dx^n \xi_{ba}(x,\mu) \cdot (\xi_{ab}(x,-\mu)U_{-\vec{m}})$$

for $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,b)$ and $\xi_{ba} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(b,a)$, where $U_{\vec{m}} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta=0,E}}(a,a) = T^{2n}$. For the remaining general case, it is given by

$$m(\xi_{bc}, \xi_{ab})(x,\rho) = \sum_{u\in\mathbb{Z}^n} \xi_{bc}(x + A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u+\rho) \cdot \xi_{ab}(x - A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), u) \ . \quad (27)$$

14

These structures together with the trace map defined by eq.(14) forms a category $\mathcal{C}^{pre}_{\theta=0,E}$ and the corresponding cyclic CDG category $\mathcal{C}_{\theta=0}$. In particular, we have $d^2 = 0$ for $d : V_{ab} \to V_{ab}$ with any pair $a, b \in \text{Ob}(\mathcal{C}_{\theta=0})$. Thus, $\mathcal{C}_{\theta=0}$ is a cyclic DG category.

For any $a, b \in \text{Ob}(\mathcal{C}_{\theta=0})$, $a \neq b$, the bases of the zero-th cohomology of V_{ab} are given by gaussians [41] (see also [6]) and called the *theta vector*, though here we are discussing the $\theta = 0$ case. We shall give examples of these theta vectors in noncommutative case $\theta \neq 0$ in the next subsection. The mirror dual $\hat{\mathbb{T}}^{2n}$ of this complex torus $T^{2n} := \mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$ is the real $2n$-dimensional torus with a symplectic structure $\omega := \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. In this mirror dual torus $\hat{\mathbb{T}}^{2n}$, a line bundle specified by A_a corresponds to an affine lagrangian submanifold L_a. Then, the intersection of L_a and L_b is a point $\hat{v}_{ab}$ on $\hat{\mathbb{T}}^{2n}$, which defines the set $\tilde{V}_{ab}$ of the infinite copies of the points on the covering space $\mathbb{C}^n$. The structure constant $C^{\mu\nu}_{abc,\rho} \in \mathbb{C}$ can be identified with the sum of the exponentials of the symplectic areas of the triangles $\tilde{v}_{ab}\tilde{v}_{bc}\tilde{v}_{ac}$ for any $\tilde{v}_{ab} \in \tilde{V}_{ab}$, $\tilde{v}_{bc} \in \tilde{V}_{bc}$ and $\tilde{v}_{ac} \in \tilde{V}_{ac}$ with respect to ω, where the triangles related by a parallel translations on the covering space are identified with each other and not overcounted (see [16]).

4.2 Noncommutative deformations of holomorphic line bundles

Let us consider a real $2n$-dimensional noncommutative torus T^{2n}_θ with its generators $U_1, \cdots, U_{2n}$ with the following relation:

$$U_i U_j = e^{-2\pi\sqrt{-1}\theta^{ij}} U_j U_i , \qquad \theta := \begin{pmatrix} \theta_1 & -\theta_2 \\ \theta_2^t & \theta_3 \end{pmatrix} .$$

Since $\theta \in Mat_{2n}(\mathbb{R})$ is antisymmetric, $\theta_1, \theta_3 \in Mat_n(\mathbb{R})$ are antisymmetric and $\theta_2 \in Mat_n(\mathbb{R})$ can be an arbitrary n by n matrix.

A Heisenberg module $E_{g,\theta}$, $g \in SO(2n, 2n, \mathbb{Z})$, on this noncommutative torus T^{2n}_θ is associated with two notions, the K_0 group element and the Chern character (see [24]). The Chern character of $E_{g,\theta}$ is defined by its constant curvature F, a skewsymmetric $2n$ by $2n$ matrix with entries in $\mathbb{R}$. On the other hand, the K_0 group element of $E_{g,\theta}$ is defined by F_0, the constant curvature of $E_{g,\theta}$ when we set $\theta = 0$. Thus, the K_0 group element is independent of the non-commutativity θ. A Heisenberg module $E_{g,\theta}$ is thought of as a noncommutative analog of a line bundle if $g \in SO(2n, 2n, \mathbb{Z})$ is of the form:

$$g = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_0 & \mathbf{1}_{2n} \end{pmatrix}$$

for a skew symmetric matrix $F_0 \in Mat_{2n}(\mathbb{Z})$. In fact, F_0 corresponds to the first Chern character of a line bundle if $\theta = 0$. Since we shall discuss noncommutative deformations of the line bundles in the previous subsection, let us consider in particular the case that F_0 is of the following form

$$F_0 = \begin{pmatrix} \mathbf{0}_n & A \\ -A & \mathbf{0}_n \end{pmatrix} ,$$

where $A \in Mat_n(\mathbb{Z})$ is a nondegenerate symmetric matrix. For the Heisenberg modules $E_{g,\theta}$ with g given as above, we shall consider noncommutative tori of the following three cases:

$$\textbf{Type } \theta_1\colon \theta_2 = \theta_3 = 0, \quad \textbf{Type } \theta_2\colon \theta_1 = \theta_3 = 0, \quad \textbf{Type } \theta_3\colon \theta_1 = \theta_2 = 0 \ .$$

In each case, the endomorphism algebra, $T_{\theta'}$, $\theta' := (\mathbf{1}_n \theta + \mathbf{0}_n)(F_0 \theta + \mathbf{1}_n)^{-1}$, turns out to be as follows. In Type θ_1 case and θ_3 case:

$$\theta = \begin{pmatrix} \theta_1 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix} , \qquad \theta = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \theta_3 \end{pmatrix} ,$$

we have $\theta' = \theta$. However, in Type θ_2 case, one obtains

$$\theta' = \begin{pmatrix} \mathbf{0}_n & -\theta_2(\mathbf{1}_n + A\theta_2)^{-1} \\ \theta_2^t(\mathbf{1}_n + A\theta_2^t)^{-1} & \mathbf{0}_n \end{pmatrix} , \qquad \theta := \begin{pmatrix} \mathbf{0}_n & -\theta_2 \\ \theta_2^t & \mathbf{0}_n \end{pmatrix} . \qquad (28)$$

Now, for the cyclic DG category $\mathcal{C}_{\theta=0}$ in the previous subsection, we construct its noncommutative deformations of three types above explicitly as CDG categories. Namely, we construct noncommutative deformations of $\mathcal{C}_{\theta=0,E}^{pre}$ in the previous subsection.

Type θ_1 A category $\mathcal{C}_{\theta,E}^{pre}$ is constructed as follows. Any object $a \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{pre})$ is associated with a pair $E_a := (E_{g_a,\theta}, \nabla_a)$. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a \in Mat_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n} \end{pmatrix} , \qquad F_{a,0} := \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} .$$

For $\xi_a \in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu) * e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} ,$$
$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i) , \qquad i = 1, \cdots, n \ .$$

Here $* : C^\infty(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is the Moyal star product ([32]) defined by

$$(f * g)(x) := f(x) e^{\frac{\sqrt{-1}}{4\pi}\overleftarrow{\partial_x}\theta_1\overrightarrow{\partial_x}} g(x) ,$$

where $\overleftarrow{\partial_x}\theta_1\overrightarrow{\partial_x} := \sum_{p,q=1}^n \overleftarrow{\partial_{x_p}}\theta_1^{pq}\overrightarrow{\partial_{x_q}}$. The generator of the endomorphism is then given by

$$(Z_i\xi_a)(x;\mu) = e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} * \xi_a(x;\mu) ,$$
$$(Z_{\bar{i}}\xi_a)(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i) .$$

A constant curvature connection $\nabla_a : E_{g_a,\theta} \otimes \mathcal{L}_\theta \to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1}, \cdots, \nabla_{a,n})^t = \begin{pmatrix} \mathbf{1}_n \\ \frac{1}{2}A_a\theta_1 & -A_a \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} ,$$

whose the constant curvature is

$$F_a = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & -A_a\theta_1 A_a \end{pmatrix} .$$

We assume that A_{ab} is nondegenerate for any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, $a \neq b$. For any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, the space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$ is defined as follows. If $a \neq b$, $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) := \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of T_{θ_a}, generated by U_i and $U_{\bar{i}}$, and the left action of T_{θ_b}, generated by Z_i and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu) * e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))} \ ,$$
$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i) \ ,$$
$$(Z_i\xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))} * \xi_{ab}(x;\mu) \ ,$$
$$(Z_{\bar{i}}\xi_{ab})(x;\mu) = \xi_{ab}(x + A_{ab}^{-1}t_i;\mu - t_i) \ .$$

If $a = b$, then $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) = T_{\theta_a} = T_{\theta_b}$ and these actions are defined by the usual product of noncommutative torus $T_{\theta_a} = T_{\theta_b}$. The constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$ is given by

$$(\nabla_1 \cdots \nabla_{2n})^t := \begin{pmatrix} \mathbf{1}_n \\ \frac{1}{2}A_{ab}^+\theta_1 & -A_{ab} \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} \ , \qquad A_{ab}^+ := A_a + A_b$$

if $a \neq b$, and if $a = b$, it is defined by the derivation ∇ of the noncommutative torus $T_{\theta_a} = T_{\theta_b}$ in eq.(18).

For any $a, b, c \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, c)$ is given as follows:

For $a = b$, it is the right action of $T_{\theta_a} = T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$.

For $b = c$, it is the left action of $T_{\theta_b} = T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$.

For $a = c$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, a) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to T_{\theta_a}$ is given by

$$m(\xi_{ba}, \xi_{ab})(x, \rho) = \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\vec{m}} \int_{\mathbb{R}^n} dx^n \xi_{ba}(x, \mu) * (\xi_{ab}(x, -\mu)U_{-\vec{m}}) \ .$$

For the remaining general case, it is given by

$$m(\xi_{bc}, \xi_{ab})(x, \rho) = \sum_{u \in \mathbb{Z}^n} \xi_{bc}(x + A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u + \rho) * \xi_{ab}(x - A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), u) \ .$$

The trace map is then given by eq.(17).

Type θ_2 A category $\mathcal{C}^{pre}_{\theta,E}$ is constructed as follows. Any object $a \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$ is associated with a pair $E_a := (E_{g_a,\theta}, \nabla_a)$, where we assume $\det(\mathbf{1}_n + \theta_2 A_a) \neq 0$, which is always satisfied if one of the entries of θ_2 is irrational. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_a\mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a \in Mat_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n} \end{pmatrix} \ , \qquad F_{a,0} = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} \ .$$

For $\xi_a \in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x;\mu) = \xi_a(x;\mu)e^{2\pi\sqrt{-1}x_i + (A_a^{-1}\mu)_i))} \ ,$$

$$(\xi_a U_{\bar{i}})(x;\mu) = \xi_a(x + (\mathbf{1}_n + \theta_2 A_a)A_a^{-1}t_i;\mu - t_i) \ .$$

The action of the generators of the endomorphism is then given by

$$(Z_i \xi_a)(x;\mu) = e^{2\pi\sqrt{-1}(((\mathbf{1}_n + \theta_2 A_a)^{-1}x)_i + (A_a^{-1}\mu)_i))}\xi_a(x;\mu) \ ,$$

$$(Z_{\bar{i}} \xi_a)(x;\mu) = \xi_a(x + A_a^{-1}t_i;\mu - t_i) \ ,$$

where the relation is

$$Z_i Z_j = e^{-2\pi\sqrt{-1}\theta_a} Z_j Z_i \ , \qquad \theta_a = \begin{pmatrix} \mathbf{0}_n & -(\mathbf{1}_n + \theta_2 A_a)^{-1}\theta_2 \\ \theta_2^t(\mathbf{1}_n + A_a\theta_2^t)^{-1} & \mathbf{0}_n \end{pmatrix} \ .$$

A constant curvature connection $\nabla_a : E_{g_a,\theta} \otimes \mathcal{L}_\theta \to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1}\cdots\nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & -(A_a^{-1} + \theta_2)^{-1} \end{pmatrix}\begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} \ ,$$

with its curvature

$$F_a = \begin{pmatrix} \mathbf{0}_n & (A_a^{-1} + \theta_2^t)^{-1} \\ -(A_a^{-1} + \theta_2)^{-1} & \mathbf{0}_n \end{pmatrix} \ .$$

We assume that A_{ab} is nondegenerate for any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, $a \neq b$.

For any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, the space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b)$ is defined as follows. If $a \neq b$, $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) :=$ $\mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of T_{θ_a}, generated by U_i and $U_{\bar{i}}$, and the left action of T_{θ_b}, generated by Z_i and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab}U_i)(x;\mu) = \xi_{ab}(x;\mu)e^{2\pi\sqrt{-1}((\mathbf{1}_n + \theta_2 A_a)^{-1}x)_i + (A_{ab}^{-1}\mu)_i))} \ ,$$

$$(\xi_{ab}U_{\bar{i}})(x;\mu) = \xi_{ab}(x + (\mathbf{1}_n + \theta_2 A_b)A_{ab}^{-1}t_i;\mu - t_i) \ ,$$

$$(Z_i \xi_{ab})(x;\mu) = e^{2\pi\sqrt{-1}(((\mathbf{1}_n + \theta_2 A_b)^{-1}x)_i + (A_{ab}^{-1}\mu)_i))}\xi_{ab}(x;\mu) \ ,$$

$$(Z_{\bar{i}} \xi_{ab})(x;\mu) = \xi_{ab}(x + (\mathbf{1}_n + \theta_2 A_a)A_{ab}^{-1}t_i;\mu - t_i) \ .$$

If $a = b$, then $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) := T_{\theta_a} = T_{\theta_b}$ and these actions are defined by the usual product of noncommutative torus $T_{\theta_a} = T_{\theta_b}$. Note that $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) = \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ is isomorphic to $E_{g_b g_a^{-1},\theta}$; for $a \neq b$, an element $\xi_{ab} \in \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$ is identified with $\xi'_{ab} \in E_{g_b g_a^{-1},\theta}$ by the following relation:

$$\xi'_{ab}(x',\mu) = \xi'_{ab}((\mathbf{1}_n + \theta_2 A_a)^{-1}x,\mu) = \xi_{ab}(x,\mu) \ .$$

A constant curvature connection $\nabla : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b)$ is given by

$$(\nabla_1\cdots\nabla_{2n})^t = \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & -(A_b^{-1} + \theta_2)^{-1} + (A_a^{-1} + \theta_2) \end{pmatrix}\begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a \neq b$, and if $a = b$, it is defined by the derivation ∇ of noncommutative torus $T_{\theta_a} = T_{\theta_b}$ in eq.(18).

18

For any $a, b, c \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$, the product $m :$ $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, c)$ is given as follows:

For $a = b$, it is the right action of $T_{\theta_a} = T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$.

For $b = c$, it is the left action of $T_{\theta_b} = T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$.

For $a = c$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, a) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to T_{\theta_a}$ is given by

$$m(\xi_{ba}, \xi_{ab})(x, \rho) = \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{\mu \in \mathbb{Z}^n / A_{ab}\mathbb{Z}^n} U_{\vec{m}} \int_{\mathbb{R}^n} dx^n \xi_{ba}(x, \mu) \cdot (\xi_{ab}(x, -\mu)U_{-\vec{m}}) \ .$$

For the remaining general case, it is given by

$m(\xi_{bc}, \xi_{ab})(x, \rho) =$
$$\sum_{u \in \mathbb{Z}^n} \xi_{bc}(x + (\mathbf{1}_n + \theta_2 A_c)A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), -u + \rho) \cdot \xi_{ab}(x - (\mathbf{1}_n + \theta_2 A_a)A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho), u) \ .$$

The trace map is then given by eq.(17).

Type θ_3 A category $\mathcal{C}^{pre}_{\theta,E}$ is constructed as follows. Any object $a \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$ is associated with a pair $E_a := (E_{g_a,\theta}, \nabla_a)$. The basic module $E_{g_a,\theta}$ is defined by

$$E_{g_a,\theta} = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n / A_a \mathbb{Z}^n))$$

for a nondegenerate symmetric matrix $A_a \in Mat_n(\mathbb{Z})$, where

$$g_a = \begin{pmatrix} \mathbf{1}_{2n} & \mathbf{0}_{2n} \\ F_{a,0} & \mathbf{1}_{2n} \end{pmatrix} \ , \qquad F_{a,0} = \begin{pmatrix} \mathbf{0}_n & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} \ .$$

For $\xi_a \in E_{g_a,\theta}$, the action of each generator is defined by

$$(\xi_a U_i)(x; \mu) = \xi_a(x; \mu)e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))} \ ,$$
$$(\xi_a U_{\bar{i}})(x; \mu) = \xi_a(x + A_a^{-1}t_i; \mu - t_i)e^{-\pi\sqrt{-1}x^t A_a \theta_3 t_i} \ , \qquad i = 1, \cdots, n \ ,$$

and the endomorphisms are generated by

$$(Z_i \xi_a)(x; \mu) = e^{2\pi\sqrt{-1}(x_i + (A_a^{-1}\mu)_i))}\xi_a(x; \mu) \ ,$$
$$(Z_{\bar{i}}\xi_a)(x; \mu) = e^{\pi\sqrt{-1}x^t A_a \theta_3 t_i}\xi_a(x + A_a^{-1}t_i; \mu - t_i) \ .$$

A constant curvature connection $\nabla_a : E_{g_a,\theta} \otimes \mathcal{L}_\theta \to E_{g_a,\theta}$ is given as

$$(\nabla_{a,1} \cdots \nabla_{a,2n})^t = \begin{pmatrix} \mathbf{1}_n & -\frac{1}{2}A_a\theta_3 A_a \\ \mathbf{0}_n & -A_a \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix} \ ,$$

with its curvature

$$F_a = \begin{pmatrix} -A_a\theta_3 A_a & A_a \\ -A_a & \mathbf{0}_n \end{pmatrix} \ .$$

We assume that A_{ab} is nondegenerate for any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, $a \neq b$.

19

For any $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta, E})$, the space $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$ is defined as follows. If $a \neq b$, $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) := \mathrm{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) = \mathcal{S}(\mathbb{R}^n \times (\mathbb{Z}^n/A_{ab}\mathbb{Z}^n))$; the right action of T_{θ_a}, generated by U_i and $U_{\bar{i}}$, and the left action of T_{θ_b}, generated by Z_i and $Z_{\bar{i}}$, are defined by

$$(\xi_{ab} U_i)(x; \mu) = \xi_{ab}(x; \mu) e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))} \ ,$$

$$(\xi_{ab} U_{\bar{i}})(x; \mu) = \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i) e^{-\pi\sqrt{-1}x^t A_{ab}\theta_3 t_i} \ ,$$

$$(Z_i \xi_{ab})(x; \mu) = e^{2\pi\sqrt{-1}(x_i + (A_{ab}^{-1}\mu)_i))} \xi_{ab}(x; \mu) \ ,$$

$$(Z_{\bar{i}} \xi_{ab})(x; \mu) = e^{\pi\sqrt{-1}x^t A_{ab}\theta_3 t_i} \xi_{ab}(x + A_{ab}^{-1} t_i; \mu - t_i) \ .$$

If $a = b$, then $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) := T_{\theta_a} = T_{\theta_b}$ and the left and right actions on it are defined by the usual product of noncommutative torus $T_{\theta_a} = T_{\theta_b}$.

A constant curvature connection $\nabla_a : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \otimes \mathcal{L}_\theta \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$ is given by

$$(\nabla_1 \cdots \nabla_{2n})^t = \begin{pmatrix} \mathbf{1}_n & -\frac{1}{2}A_{ab}^+\theta_3 A_{ab} \\ \mathbf{0}_n & -A_{ab} \end{pmatrix} \begin{pmatrix} \partial_x \\ 2\pi\sqrt{-1}x \end{pmatrix}$$

if $a \neq b$, and if $a = b$, it is defined by the usual derivation ∇ of noncommutative torus $T_{\theta_a} = T_{\theta_b}$ (18).

For any $a, b, c \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$ and $\xi_{ab} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$, $\xi_{bc} \in \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, c)$ is given as follows:

For $a = b$, it is the right action of $T_{\theta_a} = T_{\theta_b}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, c)$.

For $b = c$, it is the left action of $T_{\theta_b} = T_{\theta_c}$ on $\mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$.

For $a = c$, the product $m : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(b, a) \otimes \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to T_{\theta_a}$ is given by

$$m(\xi_{ba}, \xi_{ab})(x, \rho) = \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{\mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n} U_{\vec{m}} \int_{\mathbb{R}^n} dx^n \xi_{ba}(x, \mu) \cdot (\xi_{ab}(x, -\mu) U_{-\vec{m}}) \ .$$

For the remaining general case, it is given by

$$m(\xi_{bc}, \xi_{ab})(x, \rho) = \sum_{u \in \mathbb{Z}^n} \xi_{bc}(x', -u + \rho) \cdot \exp\left(-\pi\sqrt{-1}x'^t A_{bc}\theta_3 A_{ab}x''\right) \cdot \xi_{ab}(x'', u) \ ,$$

where $x' := x + A_{bc}^{-1}(u - A_{ab}A_{ac}^{-1}\rho)$ and $x'' := x - A_{ab}^{-1}(u - A_{ab}A_{ac}^{-1}\rho)$.

The trace map is then given by eq.(17).

By direct calculations, one obtains the followings.

Lemma 4.1 *For a fixed noncommutative parameter of type θ_1, θ_2 or θ_3,*

(Associativity): the composition m of morphisms is associative.

(Leibniz rule): the constant curvature connections on morphisms satisfy the Leibniz rule. ∎

Then, in any case of the Type θ_s, $s = 1, 2, 3$, $\mathcal{C}^{pre}_{\theta,E}$ forms a category in Definition 3.1 and then the corresponding category $\mathcal{C}$ forms a cyclic CDG category.

In particular, by looking at the condition that $d : V_{ab} \to V_{ab}$ satisfies $d^2 = 0$ explicitly, one can see the followings:

Proposition 4.2 *For Type θ_1, Type θ_2 such that $\theta_2^t = -\theta_2$ and Type θ_3, two objects $a, b \in \mathrm{Ob}(\mathcal{C})$ in the CDG category $\mathcal{C}$ forms a full sub-DG category $\mathcal{C}^{\hat{f}}$ of $\mathcal{C}$ for some $\hat{f} \in \Lambda^2 \subset V^2$ if and only if*

$$A_a \theta_s A_a = A_b \theta_s A_b , \qquad s = 1, 2, 3 \tag{29}$$

holds. ∎

Thus, we have obtained the cyclic DG categories $\mathcal{C}^{\hat{f}}$ on noncommutative tori with three types of noncommutativities θ_1, θ_2 and θ_3.

Let us calculate the compositions of the morphisms of the zero-th cohomologies $H^0(\mathcal{V}) := \oplus_{a,b \in \mathrm{Ob}(\mathcal{C}^{\hat{f}})} H^0(V_{ab})$ of these DG categories $\mathcal{C}^{\hat{f}}$. They define ring structures on $H^0(\mathcal{V})$, which are sub-rings of the full cohomologies $H^*(\mathcal{V})$. We shall observe that the ring $H^0(\mathcal{V})$ actually depends on the noncommutative parameter θ_s, as opposed to the complex one-tori case [13, 21, 33, 15]. We remark that, from a homotopy algebraic viewpoint, a DG category is homotopy equivalent to a *minimal A_∞* category. Then, by forgetting the higher compositions of the minimal A_∞ category, one obtains the ring $H^*(\mathcal{V})$. Thus, if at least the sub-ring $H^0(\mathcal{V})$ is deformed, the minimal A_∞-structure is also deformed.

Recall that $H^0(V_{ab})$ is given by $\mathrm{Ker}(d : V_{ab}^0 \to V_{ab}^1) = \cap_{i=1}^n \mathrm{Ker}(\bar{\nabla}_i : V_{ab}^0 \to V_{ab})$.

Type θ_1 For $a, b \in \mathrm{Ob}(\mathcal{C}_{\theta,E}^{pre})$, $a \neq b$, the holomorphic structure $\bar{\nabla}_i : \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b) \to \mathrm{Hom}_{\mathcal{C}_{\theta,E}^{pre}}(a, b)$ is given by

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n)^t = \left(\mathbf{1}_n + \frac{\sqrt{-1}}{2} A_{ab}^+ \theta \right) \partial_x + 2\pi A_{ab} x .$$

The cohomology $H^0(V_{ab})$ is spanned by the bases of the form

$$e_{ab}^\mu(x; \rho) := C_{ab} \delta_{[A_{ab}]_\rho^\mu} \exp\left(-\pi \left(x^t M_{ab} x \right) \right) , \qquad \mu \in \mathbb{Z}^n / A_{ab} \mathbb{Z}^n , \tag{30}$$

where $C_{ab} \in \mathbb{C}$ is an appropriate rescaling and $M_{ab} \in Mat_n(\mathbb{C})$ should be a symmetric matrix satisfying

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n) \left(\exp\left(-\pi \left(x^t M_{ab} x \right) \right) \right) = 0 . \tag{31}$$

The condition (31) turns out to be

$$-\left(\mathbf{1}_n + \frac{\sqrt{-1}}{2} A_{ab}^+ \theta_1 \right) M_{ab} + A_{ab} = 0 .$$

This M_{ab} is symmetric if and only if the condition (29) holds:

$$A_a \theta_1 A_a = A_b \theta_1 A_b ,$$

and then the explicit form of M_{ab} is given by

$$M_{ab} = A_{ab} \left(A_{ab} + \frac{\sqrt{-1}}{2} (A_a \theta_1 A_b - A_b \theta_1 A_a) \right)^{-1} A_{ab} .$$

Here the real part of $M_{ab} \in Mat_n(\mathbb{C})$ should be positive definite in order for e_{ab}^μ to exist in $H^0(V_{ab})$. Note that the part of M_{ab} is positive definite if and only if A_{ab} is positive definite. This

21

$e^{\mu}_{ab} \in H^0(V_{ab})$ in eq.(30) is a *theta vector*. Thus, for $a, b \in \mathrm{Ob}(\mathcal{C})$ such that $A_a \theta_1 A_a = A_b \theta_1 A_b$ and A_{ab} is positive definite, $\dim(H^0(V_{ab})) = \sharp(\mathbb{Z}^n/A_{ab}\mathbb{Z}^n) = \det(A_{ab})$. For the rescaling C_{ab} in eq.(30), we set

$$C_{ab} := \frac{\det(\mathbf{1}_n + \sqrt{-1}A_a\theta)^{\frac{1}{4}} \det(\mathbf{1}_n + \sqrt{-1}A_b\theta)^{\frac{1}{4}}}{\det(\mathbf{1}_n + \frac{\sqrt{-1}}{2}A^+_{ab}\theta)^{\frac{1}{2}}} \ .$$

Assume now that A_{ab} and A_{bc} are positive definite. Then we get the product formula:

$$m(e^{\mu}_{ab}, e^{\nu}_{bc}) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^{\mu}_{-u+\rho}} \delta_{[A_{bc}]^{\nu}_{u}}$$

$$\exp\left(-\pi(u - A_{bc}A^{-1}_{ac}\rho)^t \left((A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + \sqrt{-1}A_b\theta)^{-1}\right)(u - A_{bc}A^{-1}_{ac}\rho)\right) \cdot e^{\rho}_{ac} \ .$$

Note that the n by n matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + \sqrt{-1}A_b\theta)^{-1}$ is automatically symmetric due to the condition $A_a \theta_1 A_a = A_b \theta_1 A_b = A_c \theta_1 A_c$.

In this Type θ_1 case, these theta vectors $\{e^{\mu}_{ab}\}$ can be described by theta functions, where the product of two theta vectors just corresponds to the Moyal product of two theta functions [16].

Type θ_2 For $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, $a \neq b$, the holomorphic structure $\bar{\nabla}_i : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b) \to \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a, b)$ is given by

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n)^t = \partial_x + 2\pi\left((A^{-1}_b + \theta_2)^{-1} - (A^{-1}_a + \theta_2)^{-1}\right)x \ .$$

The theta vectors are of the form

$$e^{\mu}_{ab}(x, \rho) = \delta_{[A_{ab}]^{\mu}_{\rho}} \exp\left(-\pi x^t M_{ab}x\right) \ , \qquad \mu \in \mathbb{Z}^n/A_{ab}\mathbb{Z}^n \ ,$$

where $M_{ab} \in Mat_n(\mathbb{R}) \subset Mat_n(\mathbb{C})$ is given by

$$M_{ab} := (A^{-1}_b + \theta_2)^{-1} - (A^{-1}_a + \theta_2)^{-1} = (\mathbf{1}_n + A_b\theta_2)^{-1}A_{ab}(\mathbf{1}_n + \theta_2 A_a)^{-1} \ , \tag{32}$$

which should be a symmetric positive definite matrix. Then, one has $\dim(H^0(V_{ab})) = \det(A_{ab})$.

Assume that M_{ab} and M_{bc} are positive definite. Then, M_{ac} is also positive definite. The product of e^{μ}_{ab} with e^{ν}_{bc} is then

$$m(e^{\mu}_{ab}, e^{\nu}_{bc}) = \sum_{\rho \in \mathbb{Z}^n/A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^{\mu}_{-u+\rho}} \delta_{[A_{bc}]^{\nu}_{u}}$$

$$\exp\left(-\pi(u - A_{bc}A_{ac}\rho)^t((A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + A_b\theta^t_2)(\mathbf{1}_n + A_b\theta_2)^{-1})(u - A_{bc}A_{ac}\rho)\right) \cdot e^{\rho}_{ac} \ .$$

In particular, one can see that the structure constant does not depend on θ_2 if and only if *theta*$_2$ is symmetric: $\theta^t_2 = \theta_2$. This gives the reason that the structure constant of the product does not depend on the noncommutative parameter in the case of noncommutative real two-tori [13, 33, 15]. See also [23, 22], where for a complex two-tori with noncommutativity of Type θ_2 with symmetric θ_2, such structure constants are computed and checked to be independent of the noncommutativity θ_2.

When θ_2 is antisymmetric, M_{ab} in eq.(32) is symmetric if and only if $A_a \theta_2 A_a = A_b \theta_2 A_b$, where the n by n matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + A_b\theta^t_2)(\mathbf{1}_n + A_b\theta_2)^{-1} \in Mat_n(\mathbb{C})$ is symmetric:

$$(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n + A_b\theta^t_2)(\mathbf{1}_n + A_b\theta_2)^{-1} = (\mathbf{1}_n - \theta_2 A_b)^{-1}(\mathbf{1}_n - \theta^t_2 A_b)(A^{-1}_{ab} + A^{-1}_{bc}) \ .$$

Type θ_3 For $a, b \in \mathrm{Ob}(\mathcal{C}^{pre}_{\theta,E})$, $a \neq b$, the holomorphic structure $\bar{\nabla}_i : \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b) \rightarrow \mathrm{Hom}_{\mathcal{C}^{pre}_{\theta,E}}(a,b)$ is given by

$$(\bar{\nabla}_1 \cdots \bar{\nabla}_n)^t = \partial_x + 2\pi \left(\mathbf{1}_n - \frac{\sqrt{-1}}{2} A^+_{ab}\theta_3 \right) A_{ab} x \ .$$

The theta vectors are of the form

$$e^\mu_{ab}(x, \rho) = \delta_{[A_{ab}]^\mu_\rho} \exp\left(-\pi x^t M_{ab} x \right) \ , \qquad \mu \in \mathbb{Z}^n / A_{ab}\mathbb{Z}^n \ ,$$

where $M_{ab} \in Mat_n(\mathbb{C})$ is given by

$$M_{ab} := \left(\mathbf{1}_n - \frac{\sqrt{-1}}{2} A^+_{ab}\theta_3 \right) A_{ab} \ ,$$

which should be a symmetric matrix whose real part is positive definite. Here, again, the real part of M_{ab} is positive definite if and only if A_{ab} is positive definite. Then, one has $\dim(H^0(V_{ab})) = \det(A_{ab})$. The condition that M_{ab} above is symmetric is equal to

$$A^+_{ab}\theta_3 A_{ab} = (A^+_{ab}\theta_3 A_{ab})^t \ ,$$

which is in fact equivalent to $A_a\theta_3 A_a = A_b\theta_3 A_b$.

Assume now that A_{ab} and A_{bc} are positive definite. The product of two theta vectors is given by

$$m(e^\mu_{ab}, e^\nu_{bc}) = \sum_{\rho \in \mathbb{Z}^n / A_{ac}\mathbb{Z}^n} \sum_{u \in \mathbb{Z}^n} \delta_{[A_{ab}]^\mu_{-u+\rho}} \delta_{[A_{bc}]^\nu_u}$$
$$\exp\left(-\pi(u - A_{bc}A^{-1}_{ac}\rho)^t \left((A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n - \sqrt{-1}A_b\theta_3) \right) (u - A_{bc}A^{-1}_{ac}\rho) \right) \cdot e^\rho_{ac} \ .$$

One can show that the matrix $(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n - \sqrt{-1}A_b\theta_3)$ defining a quadratic form in the expression above is symmetric:

$$(A^{-1}_{ab} + A^{-1}_{bc})(\mathbf{1}_n - \sqrt{-1}A_b\theta_3) = (\mathbf{1}_n + \sqrt{-1}\theta_3 A_b)(A^{-1}_{ab} + A^{-1}_{bc}) \ .$$

Thus, we have seen that the structure constants $C^{\mu\nu}_{abc,\rho}$ depend on the noncommutative parameter θ in all these three cases.

Though in this paper we have fixed a constant curvature connection for a Heisenberg module, we can also take all the constant curvature connections on a Heisenberg module into account in a similar way as in the noncommutative complex one-tori case [13, 15]. It might also be interesting to investigate the details of the moduli space of the constant curvature connections on a Heisenberg module on T^{2n}_θ which is known to form a commutative torus T^{2n}.

We end with showing an example for the case of noncommutative complex two-torus ($n = 2$). In this case, for any fixed θ_s, $s = 1, 2, 3$, the condition eq.(29) reduces to

$$\det(A_a) = \det(A_b) \ .$$

Thus, for the objects of $\mathcal{C}^f$, one can in general have infinite number of objects. For instance, diagonal matrices $A \in Mat_n(\mathbb{Z})$ with $\det(A) = -4$ are

$$A_a = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} , \quad A_b = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} , \quad A_c = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and $A_{a'} := -A_a$, $A_{b'} := -A_b$, $A_{c'} := -A_c$. Since the zero-th cohomologies of morphisms between $\{a, b, c\}$ and $\{a', b', c'\}$ are absent, let us concentrate on the one side $\{a, b, c\}$. Then, all A_{ab}, A_{bc} and A_{ac} are positive definite, and the dimensions of the zero-th cohomologies are

$$\dim(H^0(V_{ab})) = 2 , \qquad \dim(H^0(V_{bc})) = 2 , \qquad \dim(H^0(V_{ac})) = 9 .$$

However, there exist infinite symmetric matrices $A \in Mat_n(\mathbb{Z})$ with $\det(A) = -4$, since the matrix $g^t A g$ has $\det(A) = -4$ for any $SL(2, \mathbb{Z})$ element g.

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